Tutorial Week 7.

Power series.

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

where x_0 is the centre of this series

Taylor expansion.

$$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(x_0)}{m!}(x-x_0)^m$$

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e.g. $f(x) = \frac{1}{1-x} = (1-x)^{-1}$

Sol. $f'(x) = -(1-x)^{-2}(-1) = (1-x)^{-2}$

$$f''(x) = 2\cdot(1-x)^{-3} = 2!\,(1-x)^{-3}$$

$$\vdots$$

$$f^{(n)}(x) = n!\,(1-x)^{-(n+1)}$$

Let the centre $x_0 = 0$, then

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} x^m$$

$$= \sum_{m=0}^{\infty} \frac{m!}{m!} x^m = \sum_{m=0}^{\infty} x^m$$

for $|x| < 1$. (Why ??)

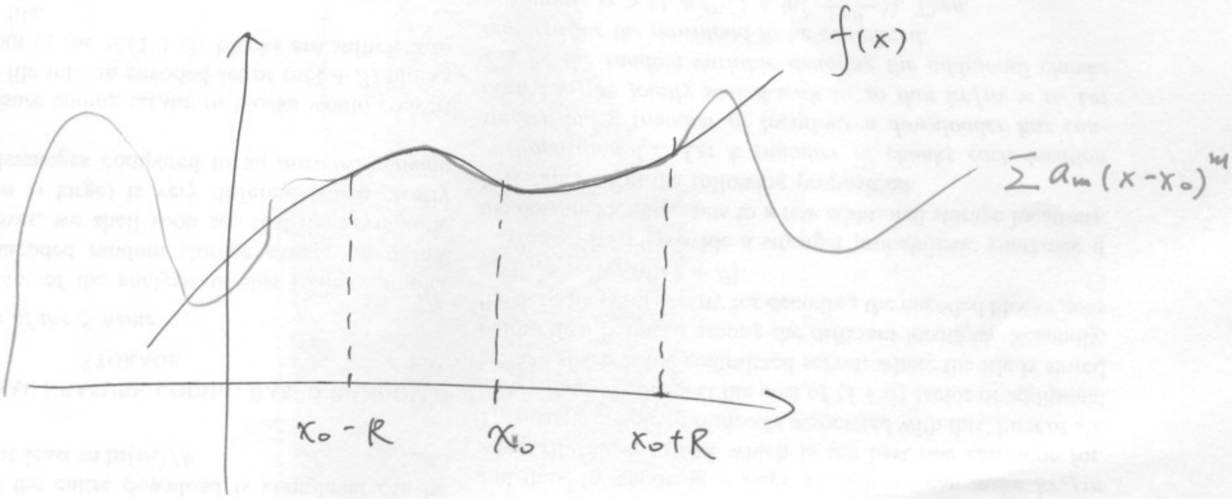
e.g. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Let R be the radius of convergence and

$$R = \frac{1}{\lim_{m\to\infty} \sqrt[m]{|a_m|}} \quad \text{or} \quad R = \frac{1}{\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right|}$$

Geometrically, the radius of convergence represents the interval that the power series converges to the function. If x_0 is the centre, then for $x \in (x_0 - R, x_0 + R)$ the power series converges to the original function.

for example.



In this case, if we write $f(x) = \sum_{m=0}^{\infty} a_m (x-x_0)^m$, the equality only true for $x \in (x_0 - R, x_0 + R)$

i.e. for $|x - x_0| < R$, $f(x) = \sum_{m=0}^{\infty} a_m (x-x_0)^m$.

e.g. $f(x) = \frac{1}{1-x}$ where centre $x_0 = 0$.

So, $f(x) = \frac{1}{1-x} = \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} x^m$

Note that $a_m = 1 \quad \forall m \in \mathbb{N}$.

So, $R = 1$.

For $f(x) = \frac{1}{a-x}$ centre at $x_0 = b \neq a$

Let $f(x) = \frac{1}{a-x} = \sum_{m=0}^{\infty} a_m (x-b)^m$.

Note that $\frac{1}{a-x} = \frac{1}{a-b-(x-b)}$

$= \frac{1}{a-b} \cdot \frac{1}{1-\frac{x-b}{a-b}}$.

Let $y = \frac{x-b}{a-b}$, then $f(x) = \frac{1}{a-b} \cdot \frac{1}{1-y}$

for $x_0 = b$, then $y_0 = 0$.

So, $\frac{1}{1-y} = \sum_{m=0}^{\infty} y^m = \sum_{m=0}^{\infty} \left(\frac{x-b}{a-b}\right)^m$

Therefore, the Taylor expansion of $f(x)$ where centre at b is $\sum_{m=0}^{\infty} \frac{1}{(a-b)^{m+1}} (x-b)^m$.

So, $a_m = \dfrac{1}{(a-b)^{m+1}}$

$$R = \frac{1}{\lim\limits_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|} = \frac{1}{\lim\limits_{m\to\infty}\frac{(a-b)^{m+1}}{(a-b)^{m+2}}} = a-b$$

Hence, $f(x) = \dfrac{1}{a-x} = \displaystyle\sum_{m=0}^{\infty} \frac{1}{(a-b)^{m+1}} (x-b)^m$

for $x \in (b-a+b,\ b+a-b)$

i.e. $x \in (2b-a,\ a)$.

Solving differential equations using power series.

example. $y'' + x^2 y = 0$

Let $y = \displaystyle\sum_{m=0}^{\infty} a_m (x-x_0)^m$,

we have $y'' = \displaystyle\sum_{m=2}^{\infty} m(m-1) a_m (x-x_0)^{m-2}$

$$\overset{n=m-2}{=} \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-x_0)^n$$

Assuming $x_0 = 0$, we have

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x)^n.$$

So, $y'' + x^2 y$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$= 2a_2 + 6a_3 x + \sum_{n=0}^{\infty} (n+4)(n+3) a_{n+4} x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2}$$

$$= 2a_2 + 6a_3 x + \sum_{n=0}^{\infty} \left[(n+4)(n+3) a_{n+4} + a_n \right] x^{n+2} = 0$$

$$\Rightarrow a_2 = 0, \quad a_3 = 0$$

$$a_{n+4} = -\frac{a_n}{(n+4)(n+3)} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

So, for $n = 4k$

$$a_{4k} = (-1)^k \prod_{s=1}^{k} \frac{1}{4s \cdot (4s-1)} a_0$$

$$a_{4k+1} = (-1)^k \prod_{s=1}^{k} \frac{1}{(4s+1) \cdot 4s} a_1 ,$$

$$a_{4k+2} = a_{4k+3} = 0. \quad (\because a_2 = a_3 = 0).$$