Tutorial Week 5.

- Second order ODE. with constant coefficient.

$$ay'' + by' + cy = r(x). \quad \text{where } a, b, c \text{ const.}$$

– homogeneous ODE. $(r(x) \equiv 0)$.

The ODE can be written into:

$$ay'' + by' + cy = 0. \qquad (1)$$

The characteristic equation of (1) is

$$a\lambda^2 + b\lambda + c = 0 \qquad (2)$$

The roots of (2) are called the characteristic of (1).

Also, assuming the characteristics λ_1, λ_2 and $\lambda_1 \neq \lambda_2$. The solution of (1) can be written as

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}, \quad C_1, C_2 \text{ const.} \qquad (3)$$

If $\lambda = \lambda_1 = \lambda_2$ the solution of (1) is

$$y = D_1 e^{\lambda x} + D_2 x e^{\lambda x}, \quad D_1, D_2 \text{ const.}$$

* IMPORTANT *.

For the case that $\lambda_1, \lambda_2 \in \mathbb{C}$., (3) can be rewritten into

$$y = e^{cx} (A \cos wx + i B \sin wx).$$

(Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.)

- Nonhomogeneous ODE. $(r(x) \neq 0)$

Steps to solve nonhomogeneous ODE:

1. Find the homogeneous solution y_h for
$$ay'' + by' + cy = 0$$

2. Guess the particular solution y_p (Using table).

3. Let $y = y_h + y_p$, y is the solution of the ODE.

Example 1.

$$y'' + 3y' - 4y = e^{2x} \qquad (4)$$

1. $y_h = c_1 e^{-4x} + c_2 e^{x}$.

2. $y_p = c_3 e^{2x}$, put y_p into (4).

$$c_3 \cdot 4 \cancel{e^{2x}} + c_3 \cdot 6 \cancel{e^{2x}} - c_3 \cdot 4 \cancel{e^{2x}} = \cancel{e^{2x}}$$

$$\cancel{4c_3} + 6c_3 \cancel{- 4c_3} = 1$$

$$\therefore c_3 = 1/6.$$

Hence, $y_p = 1/6\, e^{2x}$.

3. $y = y_h + y_p = c_1 e^{-4x} + c_2 e^{x} + 1/6\, e^{2x}$.

Example 2.

$$y'' + 3y' - 4y = e^x.$$

1. $y_h = C_1 e^{-4x} + C_2 e^x$

2. since e^x is the solution of the homogeneous ODE. the guess is

$$y_p = C_3 x e^x.$$

$$\Rightarrow \quad C_3 (e^x + xe^x)' + 3C_3 (e^x + xe^x) - 4C_3 xe^x = e^x$$

$$\Rightarrow \quad C_3 (e^x + e^x + xe^x) + 3C_3 (e^x + xe^x) - 4C_3 xe^x = e^x$$

$$\Rightarrow \quad \begin{cases} 2C_3 + 3C_3 = 1 \\ C_3 + 3C_3 - 4C_3 = 0 \end{cases} \Rightarrow \quad C_3 = 1/5.$$

Hence, $y_p = 1/5\, x e^x$.

Therefore, $y = y_h + y_p = c_1 e^{-4x} + c_2 e^x + 1/5\, x e^x$.

:ample 3.

$$y'' + 3y' - 4y = e^{2x} + 2x + 3.$$

1. $y_h = c_1 e^{-4x} + c_2 e^{x}.$

2. Since $r(x) = e^{2x} + 2x + 3$, let

   $r_1(x) = e^{2x}$, $r_2(x) = 2x + 3$.

   Let y_{p_1} be the particular solution according to $r_1(x)$ and y_{p_2} for the $r_2(x)$.

   So, $y_{p_1} = \frac{1}{6} e^{2x}$ (by example 1)

   $y_{p_2} = ax + b$

   $\Rightarrow 0 + 3b - 4ax - 4b = 2x + 3$

   $\Rightarrow \begin{cases} -4a = 2 \\ 3b - 4b = 3 \end{cases} \Rightarrow \begin{cases} a = -\frac{1}{2} \\ b = -3. \end{cases}$

   $\therefore y_{p_2} = -\frac{1}{2}x - 3$

   Therefore $y_p = y_{p_1} + y_{p_2}$

   $= \frac{1}{6} e^{2x} - \frac{1}{2}x - 3$

So, $y = y_h + y_p$

$= c_1 e^{-4x} + c_2 e^{x} + \frac{1}{6} e^{2x} - \frac{1}{2}x - 3.$

· Linear (in) dependence.

Def: Let $f_1, f_2, \cdots, f_n$ be functions of x, $\{f_i\}_{i=1}^{n}$ is <u>linearly independent</u> if the following is true:

Let $a_1, a_2, \cdots, a_n$ be constants

$$a_1 f_1 + a_2 f_2 + \cdots + a_n f_n \equiv 0$$

implies $a_i = 0 \ \forall i$

e.g. $\{1, x, x^2, x^3, \cdots\}$ is linearly independent.

$\{\sin x, \cos x\}$ is linearly independent.

$\{\cos x, \cos 2x, \cos 3x, \cdots\}$ is linearly independent

$\{\cdots e^{-2x}, e^{-x}, 1, e^{x}, e^{2x}, \cdots\}$ is linearly independent.

Def: If $f_1, f_2, \cdots, f_n$ is not linearly independent, then it is linear dependent.

e.g. $\{1, x, 2x+3\}$ is linearly dependent.

$\{1, e^{ix}, \cos x + i\sin x\}$ is linearly dependent.